FUNCTIONAL EQUATIONS AND THE GENERALISED ELLIPTIC GENUS

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Abstract. We give a new derivation and characterisation of the generalised elliptic genus of Krichever-Höhn by means of a functional equation.

Introduction

Functional equations provide a common thread to several investigations in mathematics and physics: our focus in this article will particularly be on the areas of topology and integrable systems where it is still unclear whether the threads before us form part of a greater fabric. In topology the German and Russian schools applied functional equations powerfully to formal group laws and genera [8, 11, 12, 13, 16, 20, 21, 22, 23]. They have arisen in the study of integrable systems in several different ways. F. Calogero, whom we honour in this volume, instigated in [18] a new use of functional equations in the study of integrable systems that is relevant here.

The modern approach to integrable systems is to utilise a Lax pair. Calogero in [18], by assuming a particular ansatz for a Lax pair, reduced the consistency of the Lax pair to a functional equation and algebraic constraints. In this way he discovered the elliptic Calogero-Moser model. Similarly, by assuming an ansatz for a realisation of the generators of the Poincaré algebra, Ruijsenaars and Schneider [30] reduced the ensuing commutation relations to that of a functional equation. The Ruijsenaars-Schneider model which results from one solution to this functional equation is also integrable. For the Ruijsenaars-Schneider systems Bruschi and Calogero constructed a Lax pair, again by means of an ansatz and consequent functional equation [9, 10]. (The general solutions to the functional equations of Ruijsenaars and Schneider have now been constructed [7, 17], but it is still open whether the resulting models are completely integrable.) Later Braden and Buchstaber generalised these various Lax pair ansätze [5] and encountered a rather ubiquitous functional equation [6] that includes many functional equations arising in both cohomological computations and integrable systems. We will return to this functional equation in due course but what is of interest at this stage is that the same equations arise in both settings. This may reflect something deeper. String theory physics allows some topology changes (such as flops) [1, 34] and physical quantities such as the partition function should reflect this invariance; invariance under classical flops characterises

the elliptic genus [31]. The authors of [24] draw connections between the complex cobordism ring and conformal field theory. Certainly the Baker-Akhiezer functions associated to the integrable systems satisfy addition formulae [14, 15] and reflect the underlying algebraic geometry [19].

The present article aims to provide a new derivation of the generalised elliptic genus of Krichever-Höhn by means of a functional equation encountered in the study of integrable systems. In the first section we will review equivariant genera of loop spaces. The following section derives the relevant functional equation which we then solve in the final section. Various remarks will be made enroute that relate this approach to existing derivations.

1. Equivariant Genera of the Loop Space

Motivated by the problem of obtaining left–right asymmetric fermions in a Kaluza–Klein theory Witten in [32] suggested the study of a special twisted Dirac operator on closed spin manifolds equipped with a smooth S^1-action. Witten conjectured that the character–valued index of such a twisted operator is in fact a constant and that the genus of a manifold corresponding to this Dirac operator possesses a rigidity property.

To break the conjecture into simpler pieces, Landweber posed a problem on computation of a special ideal in the bordism ring of semifree S^1-actions on spin manifolds. As a tool for the solution to this problem Ochanine [29] introduced an elliptic genus

$$Q(x) = \frac{1}{2}\frac{x}{\tanh(x/2)} \cdot \prod_{n=1}^{\infty}\left(\frac{(1+q^n\mathrm{e}^{x})(1+q^n\mathrm{e}^{-x})}{(1-q^n\mathrm{e}^{x})(1-q^n\mathrm{e}^{-x})}\cdot\frac{(1-q^n)^2}{(1+q^n)^2}\right).$$

Meanwhile Witten [33] gave an informal approach to his own conjecture by computing the equivariant signature of the space of smooth loops $\mathcal{L}M$ on the manifold M and discovered that the genus obtained is (up to a constant) equal to the Ochanine genus.

Although the equivariant signature of the loop space is not a well defined object in algebraic topology, the formal properties of the genus coming out of this procedure have many nice features. In particular, the general methods of the theory of elliptic operators and fixed point theorems [2, 3] allowed Witten to confirm his earlier conjecture in the reformulation that the elliptic genus, being formally equal to the index of a Dirac type operator on the loop space, is rigid. This statement was rigorously proved by Taubes [4].

Following Witten's general scheme one can calculate other well known genera of the loop space. Consider the Hirzebruch χ_y-genus given by power series:

$$R(x) = \frac{x\left(1+y\mathrm{e}^{-x(1+y)}\right)}{1-\mathrm{e}^{-x(1+y)}}.$$

Let M^{2n} be a stable almost complex manifold. Consider the canonical S^1-action on the loop space $\mathcal{L}M^{2n}$ induced from the standard S^1-action on

the parameters:

$$g : S^1 \times \mathcal{L}M^{2n} \to \mathcal{L}M^{2n}, \qquad g(z, \gamma(t)) = \gamma(zt), \qquad z, t \in S^1, \qquad \gamma : S^1 \to M^{2n}.$$

The fixed point set of this action consists of constant loops only and, therefore, is equal to M^{2n}. The explicit form of the restriction of the tangent bundle $T(\mathcal{L}M^{2n})$ to the loop space $\mathcal{L}M^{2n}$ on $M^{2n} \subset \mathcal{L}M^{2n}$ at a point $p \in M^{2n}$ is given by

$$T_p(\mathcal{L}M^{2n}) \cong \Gamma(S^1 \times T_pM^{2n}) = \mathcal{L}(T_pM^{2n}),$$

where $\Gamma(S^1 \times T_pM^{2n})$ is a space of sections of the bundle $S^1 \times T_pM^{2n} \to S^1$. From here the decomposition of $T(\mathcal{L}M^{2n})|_{M^{2n}}$ into eigenspaces with respect to the S^1-action $g : S^1 \times \mathcal{L}M^{2n} \to \mathcal{L}M^{2n}$ is

$$T(\mathcal{L}M^{2n})\big|_{M^{2n}} = \sum_{k=-\infty}^{\infty} q^k TM^{2n}, \tag{1}$$

where $q = e^{2\pi i t}$ acts on the k-th Fourier coefficient of the loop $\gamma : S^1 \to T_p(M^{2n})$ as multiplication by q^k (see details, for example, in [21]). The Atiyah–Bott fixed point theorem says that a genus ϕ of an almost complex manifold X with a compatible circle action is equal to the sum over all connected components X_s of the fixed point set of expressions

$$\frac{\phi\left(T(X)|_{X_s}\right)}{e(\nu_s)},$$

where $e(\nu_s)$ is the Euler class of the normal bundle ν_s of the embedding $X_s \subset X$, $s = 1, 2, \ldots$. Applying it to the decomposition into eigenspaces (1) and $\phi = \chi_y$ we obtain formally

$$\chi_y(\mathcal{L}M^{2n}) = \langle \prod_{i=1}^{m} \left(\frac{x_i\left(1 + y\mathrm{e}^{-x_i(1+y)}\right)}{1 - \mathrm{e}^{-x_i(1+y)}} \prod_{k=1}^{\infty} \frac{1 + y\tilde{q}^k \mathrm{e}^{-x_i(1+y)}}{1 - \tilde{q}^k \mathrm{e}^{-x_i(1+y)}} \cdot \frac{1 + y\tilde{q}^{-k} \mathrm{e}^{-x_i(1+y)}}{1 - \tilde{q}^{-k} \mathrm{e}^{-x_i(1+y)}} \right), [M^{2n}] \rangle, \tag{2}$$

where $\tilde{q}$ is now a formal parameter corresponding to the generator of $H^*(CP(\infty), \mathbb{Q})$. As the expression on the right hand side of (2) is not convergent, we rearrange terms in such a way that we may rewrite

$$\frac{1 + y\tilde{q}^{-k} \mathrm{e}^{-x_i(1+y)}}{1 - \tilde{q}^{-k} \mathrm{e}^{-x_i(1+y)}} = \frac{\tilde{q}^k \mathrm{e}^{x_i(1+y)} + y}{\tilde{q}^k \mathrm{e}^{x_i(1+y)} - 1}.$$

Normalising the last expression by $-y^{-1}$ we arrive as in [21, 23] to

Definition 1.1. *For a stable almost complex manifold M^{2n} the equivariant χ_y-genus of the loop space $\mathcal{L}M^{2n}$ is defined up to a normalisation by*

$$\chi_y(\mathcal{L}M^{2n}) = \langle \prod_{i=1}^{m} \left(\frac{x_i\left(1 + y\mathrm{e}^{-x_i(1+y)}\right)}{1 - \mathrm{e}^{-x_i(1+y)}} \prod_{k=1}^{\infty} \frac{1 + y\tilde{q}^k \mathrm{e}^{-x_i(1+y)}}{1 - \tilde{q}^k \mathrm{e}^{-x_i(1+y)}} \cdot \frac{1 + y^{-1}\tilde{q}^k \mathrm{e}^{x_i(1+y)}}{1 - \tilde{q}^k \mathrm{e}^{x_i(1+y)}} \right), [M^{2n}] \rangle, \tag{3}$$

for $\tilde{q} = \mathrm{e}^{2\pi i \tau}$, $\mathrm{Im}\tau > 0$.

Definition 1.1 describes a genus of stable almost complex manifolds which is, up to a constant, equal to the Krichever genus [25] that was discovered as a particularly elegant example of the theory of Conner–Floyd equations in complex cobordism for circle actions [26, 27, 28]. The theory of Conner–Floyd equations allows one to deduce powerful restrictions on the cobordism classes of stable complex manifolds with compatible circle actions by means of the theory of complex analytic functions. In particular, the rigidity property of a genus is equivalent to a convergence of special analytic expressions associated with the fixed point data of the circle action on the manifold. The Krichever genus also has a rigidity property but for a class of manifolds equipped with an SU-structure.

Following general techniques one can relate the question of the rigidity of any genus to the question of a multiplicative property for some fibre bundles. The multiplicative property of a genus g for the fibre bundle $p : E \xrightarrow{F} B$ with smooth fibre and base says that

$$g(E) = g(F) \cdot g(B).$$

It was shown by Höhn that the only genus ϕ that satisfies the multiplicative property with respect to the fibre bundles whose fibres admit an SU-structure is the Krichever genus. A particular example of such bundles plays a crucial role in the work of Totaro [31] who proved that the Krichever genus is the only genus preserved by flops. The idea of Totaro was that the difference (in the complex bordism ring of a point) of two complex manifolds equivalent via a flop is bordant to a manifold E, which is fibred over a certain complex manifold, and whose fibre is $CP(3)$, equipped with a fake stable almost complex structure that admits an SU-structure. Using the fact that the complex bordism ring Ω_*^U has no torsion and that the flop is a symmetric operation it was concluded in [31] that ϕ-genus of $CP(3)$ with such a stable almost complex structure is zero. Thus, from the multiplicative property

$$\phi(E) = \phi(\overline{CP}(3))\,\phi(B) \tag{4}$$

it follows that two manifolds equivalent via a flop have the same ϕ genus. The proof in [31], that there is no other genus preserved by flops (or equivalently, satisfying $\phi(E) = 0$ for any fibre bundle $E \xrightarrow{\overline{CP}(3)} B$), consists of an estimate on the size of the quotient of $\Omega_*^U \times \mathbb{Q}$ by bordism classes of the differences between manifolds equivalent via flops.

In the next section we rewrite the multiplicative property (4) in terms of a functional equation for the generating function of ϕ. In the final section we will solve this functional equation, giving a new derivation that ϕ is the Krichever-Höhn genus.

2. The Functional equation

Let M^{2n} be a stable almost complex compact manifold without boundary. Consider complex vector bundles ξ and η over M^{2n} of complex dimension 2. Let $CP(\xi\oplus\eta)$ be the complex projectivization of the Whitney sum $\xi\oplus\eta$, that is an associated fibre bundle over M^{2n} with fibre $CP(3)$. We introduce a stable almost complex structure on $CP(\xi\oplus\eta)$ in the following way. Observe that

$$T(CP(\xi\oplus\eta)) \overset{\mathbb{R}}{\cong} \tau_F(CP(\xi\oplus\eta)) \oplus p^*T(M^{2n}),$$

where $p : CP(\xi\oplus\eta) \to M^{2n}$ is the projection, $\tau_F = \tau_F(CP(\xi\oplus\eta))$ is the bundle of tangents along the fibre, and as usual, $T(X)$ denotes the real tangent bundle to a manifold X. It is well known that for the complex projectivization of any complex vector bundle η

$$\tau_F(CP(\eta)) \overset{\mathbb{R}}{\cong} \mathrm{Hom}_{\mathbb{C}}(\eta(1), \eta^{\perp}) \cong \eta^*(1)\otimes\eta^{\perp}, \tag{5}$$

where $\eta(1)$ is the tautological vector bundle over $CP(\eta)$, $\eta^*(1)$ is conjugate to $\eta(1)$, and $\eta^{\perp}$ is its orthogonal complement in $p^*\eta$:

$$\eta(1)\oplus\eta^{\perp} \overset{\mathbb{C}}{\cong} p^*\eta.$$

Adding the trivial complex line bundle $[1]_{\mathbb{C}}$ to the left hand side of (5) we obtain

$$\begin{aligned}\tau_F(CP(\xi\oplus\eta))\oplus[1]_{\mathbb{C}} &\overset{\mathbb{R}}{\cong} \mathrm{Hom}_{\mathbb{C}}(\eta(1),\eta^{\perp})\oplus\mathrm{Hom}_{\mathbb{C}}(\eta(1),\eta(1))\\ &\overset{\mathbb{C}}{\cong} \mathrm{Hom}_{\mathbb{C}}(\eta(1), p^*(\xi\oplus\eta))\\ &\overset{\mathbb{C}}{\cong} \eta^*(1)\otimes p^*\xi\oplus\eta^*(1)\otimes p^*\eta.\end{aligned}$$

Let us equip the bundle $\tau_F(CP(\xi\oplus\eta))$ with a stable almost complex structure in the following way

$$\tau_F(CP(\xi\oplus\eta))\oplus[1]_{\mathbb{C}} \overset{\mathbb{C}}{\cong} \eta^*(1)\otimes p^*\xi\oplus(\eta^*(1)\otimes p^*\eta)^* \overset{\mathbb{C}}{\cong} \eta^*(1)\otimes p^*\xi\oplus\eta(1)\otimes p^*\eta^*. \tag{6}$$

We define the stable almost complex structure on the projectivization of $\xi\oplus\eta$ as the sum of the complex structure (6) and the standard complex structure in $p^*T(M^{2n})$. To underline that this complex structure is not the usual one, we denote the total space of the projectivization with such a stable almost complex structure by $\overline{CP}(\xi\oplus\eta)$.

Theorem 2.1. *A complex cobordism genus $\phi : \Omega_*^U \to \mathbb{Q}$ satisfies the multiplicative property*

$$\phi(\overline{CP}(\xi\oplus\eta)) = \phi(\overline{CP}(3))\,\phi(M^{2n})$$

if and only if its generating power series $f(x) = 1 + \alpha_1 x + \alpha_2 x^2 + \dots$ is a solution of the following functional equation:

(7)
$$\lambda = \frac{f(x_2-x_1)}{x_2-x_1}\cdot\frac{f(x_1-y_1)}{x_1-y_1}\cdot\frac{f(x_1-y_2)}{x_1-y_2} + \frac{f(x_1-x_2)}{x_1-x_2}\cdot\frac{f(x_2-y_1)}{x_2-y_1}\cdot\frac{f(x_2-y_2)}{x_2-y_2}$$
$$-\frac{f(x_1-y_1)}{x_1-y_1}\cdot\frac{f(x_2-y_1)}{x_2-y_1}\cdot\frac{f(y_1-y_2)}{y_1-y_2} - \frac{f(x_1-y_2)}{x_1-y_2}\cdot\frac{f(x_2-y_2)}{x_2-y_2}\cdot\frac{f(y_2-y_1)}{y_2-y_1}$$

for some constant λ.

Proof. By the same arguments as in [21] it is sufficient to show that under the Gysin map $p_! : H^*(\overline{CP}(\xi\oplus\eta),\mathbb{Q}) \to H^{*-6}(M^{2n},\mathbb{Q})$ the cohomology class $f(\tau_F) = f(\gamma_1)\cdots f(\gamma_4)$ (where γ_i, $i = 1,\dots 4$, are the Chern roots of τ_F) is mapped into $H^0(M^{2n},\mathbb{Q})$. To check it in our case we can consider $BT^4 = CP(\infty)\times CP(\infty)\times CP(\infty)\times CP(\infty)$ as a base space instead of the manifold M^{2n}, and we can put

$$\xi \cong \eta_1\oplus\eta_2, \qquad \eta\cong\eta_3\oplus\eta_4,$$

where $\eta_1,\dots,\eta_4$ are the tautological line bundles over the corresponding factors in BT^4. To calculate the Gysin map in this situation we can use standard techniques from fixed point theory. The bundle $CP(\eta_1\oplus\dots\oplus\eta_4)$ has four section $s_1,\dots,s_4$ which are in one-to-one correspondence with the summands in $\eta_1\oplus\dots\oplus\eta_4$. Let us denote the first Chern class of η_i by ϵ_i. Now for rational cohomology the complex structure (6) induces the standard orientation of the fibre. Thus we obtain

$$p_!(f(\gamma_1)\cdots f(\gamma_4)) = \sum_{i=1}^{4}\frac{s_i^*(f(\gamma_1)\cdots f(\gamma_4))}{\prod_{j\neq i}(\epsilon_j-\epsilon_i)}. \tag{8}$$

Using the explicit form of the complex structure (6) in τ_F we derive:

$$s_1^*(\tau_F) = \eta_1^*\otimes(\eta_1\oplus\eta_2)\oplus\eta_1\otimes(\eta_3^*\oplus\eta_4^*) \overset{\mathbb{C}}{\cong} [1]_{\mathbb{C}}\oplus\eta_1^*\otimes\eta_2\oplus\eta_1\otimes\eta_3^*\oplus\eta_1\otimes\eta_4^*;$$
$$s_2^*(\tau_F) = \eta_2^*\otimes(\eta_1\oplus\eta_2)\oplus\eta_2\otimes(\eta_3^*\oplus\eta_4^*) \overset{\mathbb{C}}{\cong} \eta_2^*\otimes\eta_1\oplus[1]_{\mathbb{C}}\oplus\eta_2\otimes\eta_3^*\oplus\eta_2\otimes\eta_4^*;$$
$$s_3^*(\tau_F) = \eta_3^*\otimes(\eta_1\oplus\eta_2)\oplus\eta_3\otimes(\eta_3^*\oplus\eta_4^*) \overset{\mathbb{C}}{\cong} \eta_3^*\otimes\eta_1\oplus\eta_3^*\otimes\eta_2\oplus[1]_{\mathbb{C}}\oplus\eta_3\otimes\eta_4^*;$$
$$s_4^*(\tau_F) = \eta_4^*\otimes(\eta_1\oplus\eta_2)\oplus\eta_4\otimes(\eta_3^*\oplus\eta_4^*) \overset{\mathbb{C}}{\cong} \eta_4^*\otimes\eta_1\oplus\eta_4^*\otimes\eta_2\oplus\eta_4\otimes\eta_3^*\oplus[1]_{\mathbb{C}}.$$

Because $f([1]_{\mathbb{C}}) = 1$ and $f(c_1(\eta_i^*\otimes\eta_j)) = f(\epsilon_j-\epsilon_i)$ we deduce the following explicit formulae for the restrictions on the sections s_j, $j = 1,2,3,4$, of $CP(\eta_1\oplus\eta_2\oplus\eta_3\oplus\eta_4)$:

$$s_1^*(f(\gamma_1)\cdots f(\gamma_4)) = f(\epsilon_2-\epsilon_1)f(\epsilon_1-\epsilon_3)f(\epsilon_1-\epsilon_4);$$
$$s_2^*(f(\gamma_1)\cdots f(\gamma_4)) = f(\epsilon_1-\epsilon_2)f(\epsilon_2-\epsilon_3)f(\epsilon_2-\epsilon_4);$$
$$s_3^*(f(\gamma_1)\cdots f(\gamma_4)) = f(\epsilon_1-\epsilon_3)f(\epsilon_2-\epsilon_3)f(\epsilon_3-\epsilon_4);$$
$$s_4^*(f(\gamma_1)\cdots f(\gamma_4)) = f(\epsilon_1-\epsilon_4)f(\epsilon_2-\epsilon_4)f(\epsilon_4-\epsilon_3).$$

Finally from (8), the condition $p_!(f(\gamma_1)\cdots f(\gamma_4)) \in H^0(BT^4,\mathbb{Q})$ is equivalent to the functional equation (7). $\square$

3. Solution of the Functional equation

We shall now obtain the solution to our functional equation.

Theorem 3.1. *Let $g(x) = f(x)/x$. The general analytic solution to (7) with expansion $g(x) = 1/x + \alpha_1 + \alpha_2 x + \dots$ is given by the Krichever-Höhn elliptic genus*

$$g(x) = e^{\mu x}\,\frac{\sigma(\nu - x)}{\sigma(\nu)\,\sigma(x)}.$$

Here $\sigma(x) = \sigma(x|\omega,\omega')$ is the Weierstrass sigma function which may alternately be expressed in terms of the Jacobi theta function θ_1 as $\sigma(x|\omega,\omega') = \frac{2\omega}{\pi}\exp\left[\frac{\eta x^2}{2\omega}\right]\theta_1\left(\frac{\pi x}{2\omega}|\frac{\omega'}{\omega}\right)/\theta_1'$. Moreover, the only possible value for λ in (7) is zero.

A particular case of this will be the Ochanine genus, when $g(x)$ is odd (which corresponds to ν being a half-period), and the functional equation becomes that studied by Hirzebruch [21]. The connection with (3) is made using the Jacobi triple product formula

$$\frac{\theta_1\left(\frac{ix}{2}\,|\frac{\omega'}{\omega}\right)}{\theta_1'\left(0\,|\frac{\omega'}{\omega}\right)} = i\,\sinh(x/2)\prod_{k=1}^{\infty}\frac{(1-\bar q^{2k}e^{x})(1-\bar q^{2k}e^{-x})}{(1-\bar q^{2k})^2}, \qquad \bar q = \exp\left(i\pi\frac{\omega'}{\omega}\right).$$

Then with $\mu = \eta\nu/\omega$ we have

$$\begin{aligned} g_{\mu=\eta\nu/\omega}\left(\frac{i\omega x}{\pi}\right) &= \frac{\theta_1'\left(0\,|\frac{\omega'}{\omega}\right)}{2i\theta_1\left(\frac{i\nu}{2}\,|\frac{\omega'}{\omega}\right)}\frac{\theta_1\left(i[x-\nu]/2\,|\frac{\omega'}{\omega}\right)}{\theta_1\left(ix/2\,|\frac{\omega'}{\omega}\right)} \\ &= \frac{\theta_1'\left(0\,|\frac{\omega'}{\omega}\right)}{2i\theta_1\left(\frac{i\nu}{2}\,|\frac{\omega'}{\omega}\right)}\frac{\sinh([x-\nu]/2)}{\sinh(x/2)}\prod_{k=1}^{\infty}\frac{(1-\bar q^{2k}e^{x-\nu})(1-\bar q^{2k}e^{-x+\nu})}{(1-\bar q^{2k}e^{x})(1-\bar q^{2k}e^{-x})}, \end{aligned}$$

which yields (3) up to a normalisation upon setting $\tilde q = \bar q^2$ and $-y = \exp(-\nu)$.

The strategy of our proof will be to first show that (7) is a particular example of the the more general equation

$$\phi_1(x+y) = \frac{\begin{vmatrix}\phi_2(x) & \phi_2(y)\\ \phi_3(x) & \phi_3(y)\end{vmatrix}}{\begin{vmatrix}\phi_4(x) & \phi_4(y)\\ \phi_5(x) & \phi_5(y)\end{vmatrix}} \tag{9}$$

studied by Braden and Buchstaber [6]. This equation includes many functional equations of cohomological interest. The cited work in fact provides

a constructive method of solution we shall utilise. The general analytic solution of (9) is, up to symmetries, given by

$$\phi_1(x) = \frac{\Phi(x;\nu_1)}{\Phi(x;\nu_2)}, \quad \begin{pmatrix}\phi_2(x)\\ \phi_3(x)\end{pmatrix} = \begin{pmatrix}\Phi(x;\nu_1)\\ \Phi'(x;\nu_1)\end{pmatrix} \quad \text{and} \quad \begin{pmatrix}\phi_4(x)\\ \phi_5(x)\end{pmatrix} = \begin{pmatrix}\Phi(x;\nu_2)\\ \Phi'(x;\nu_2)\end{pmatrix},$$

where

$$\Phi(x;\nu) \equiv \frac{\sigma(\nu - x)}{\sigma(\nu)\sigma(x)}\, e^{\zeta(\nu)x}. \tag{10}$$

Here $\zeta(x) = \frac{\sigma(x)'}{\sigma(x)}$ is the Weierstrass zeta function. The parameters appearing in the solution are determined as follows. Suppose x_0 is a generic point for (9). Then (for $k = 1, 2$) we have that

(11)

$$\partial_y \ln \left| \begin{array}{cc} \phi_{2k}(x+x_0) & \phi_{2k}(y+x_0) \\ \phi_{2k+1}(x+x_0) & \phi_{2k+1}(y+x_0) \end{array} \right| \Bigg|_{y=0} = \zeta(\nu_k) - \zeta(x) - \zeta(\nu_k - x) - \lambda_k,$$

$$= -\frac{1}{x} - \lambda_k + \sum_{l=0} F_l\, \frac{x^{l+1}}{(l+1)!}.$$

The Laurent expansion determines the parameters g_1, g_2 (which are the same for both $k = 1, 2$) characterising the elliptic functions of (10) by

$$g_2 = \frac{5}{3}\left(F_2 + 6F_0^2\right), \qquad g_3 = 6F_0^3 - F_1^2 + \frac{5}{3}F_0F_2,$$

and the parameters ν_k via $F_0 = -\wp(\nu_k)$. Here $\wp(x) = -\zeta'(x)$ is the Weierstrass elliptic $\wp$-function with periods 2ω, $2\omega'$ that satisfies the differential equation $\wp'(x)^2 = 4\wp(x)^3 - g_2\wp(x) - g_3$.

Proof. Upon setting $g(x) = f(x)/x$ equation (7) may be rewritten as

$$\begin{aligned}\lambda &= g(x_2 - x_1)\, g(x_1 - y_1)\, g(x_1 - y_2) + g(x_1 - x_2)\, g(x_2 - y_1)\, g(x_2 - y_2) \\ &\quad - g(x_1 - y_1)\, g(x_2 - y_1)\, g(y_1 - y_2) - g(x_1 - y_2)\, g(x_2 - y_2)\, g(y_2 - y_1), \end{aligned}$$

(12)

$$\begin{aligned} &= g(-a)\, g(a+b)\, g(a+b+c) + g(a)\, g(b)\, g(b+c) \\ &\quad - g(a+b)\, g(b)\, g(c) - g(a+b+c)\, g(b+c)\, g(-c), \end{aligned}$$

where $x_1 - x_2 = a$, $x_2 - y_1 = b$ and $y_1 - y_2 = c$.

First observe $\lambda = 0$. This may be seen by substituting $g(x) = 1/x + \alpha_1 + \alpha_2 x + \ldots$ into (12) and determining the constant term. Then from the $b = 0$ pole term of (12) we find that

$$0 = g(a+c)\,[g(a)g(-a) - g(c)g(-c)] + g(a)g'(c) - g'(a)g(c). \tag{13}$$

This equation is then of the form (9):

$$g(a+c)=\frac{\begin{vmatrix} g(a) & g(c) \\ g'(a) & g'(c) \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ g(a)g(-a) & g(c)g(-c) \end{vmatrix}}. \tag{14}$$

We shall now employ the constructive techniques of [6]. Applying (11) to the denominator of (14) yields

$$\frac{g'(x_0)g(-x_0)-g(x_0)g'(-x_0)}{g(x_0)g(-x_0)-g(x+x_0)g(-x-x_0)}=\zeta(\nu_2)-\zeta(x)-\zeta(\nu_2-x)-\lambda_2. \tag{15}$$

Now the left-hand-side has poles at $x=0$, corresponding to $\zeta(x)=1/x$ on the right-hand-side, and a further pole at $x=-2x_0$. From this we deduce that $\nu_2=-2x_0$. Also, from the pole in $g(x+x_0)$ as $x\to -x_0$ we obtain

$$0=\zeta(-2x_0)-\zeta(-x_0)-\zeta(-x_0)-\lambda_2$$

and so $\lambda_2=2\,\zeta(x_0)-\zeta(2x_0)$. Upon using standard elliptic function identities we may rewrite (15) as

$$\frac{g(x+x_0)g(-x-x_0)-g(x_0)g(-x_0)}{g'(x_0)g(-x_0)-g(x_0)g'(-x_0)}=[\wp(x_0)-\wp(x+x_0)]\,\frac{\sigma^4(x_0)}{\sigma(2x_0)}=-\frac{\wp(x_0)-\wp(x+x_0)}{\wp'(x_0)}.$$

Comparison of the $1/(x+x_0)^2$ pole terms in this equation enables us to deduce that

$$g'(x_0)g(-x_0)-g(x_0)g'(-x_0)=-\wp'(x_0),$$

and so

$$g(x_0)g(-x_0)=\wp(\nu)-\wp(x_0)=\frac{\sigma(-\nu+x_0)\,\sigma(\nu+x_0)}{\sigma^2(\nu)\,\sigma^2(x_0)}.$$

Thus we may write

$$g(x)=h(x)\,\frac{\sigma(\nu-x)}{\sigma(\nu)\,\sigma(x)}$$

where

$$h(x)\,h(-x)=1. \tag{16}$$

It will be convenient to express h as

$$h(x)=e^{\psi(x)+\zeta(\nu)x},$$

and so $g(x)=e^{\psi(x)}\,\Phi(x;\nu)$. We deduce that ψ is an odd function from (16).

Thus far we have only derived constraints from the denominator of (14). Substituting our expression for g into (13) now yields

$$\begin{aligned} g(a+c)\,[g(c)g(-c)-g(a)g(-a)] &= e^{\psi(a+c)}\,\Phi(a+c;\nu)\,[\wp(a)-\wp(c)] \\ &= \begin{vmatrix} g(a) & g(c) \\ g'(a) & g'(c) \end{vmatrix} \\ &= e^{\psi(a)+\psi(c)}\,\Phi(a+c;\nu)\,[\wp(a)-\wp(c)] \\ &\quad + \left[\psi'(c)-\psi'(a)\right] e^{\psi(a)+\psi(c)}\,\Phi(a;\nu)\,\Phi(c;\nu). \end{aligned}$$

Again using standard elliptic function identities this may be rewritten as

$$\begin{aligned} e^{\psi(a+c)-\psi(a)-\psi(c)} &= 1+\left[\psi'(c)-\psi'(a)\right]\frac{\Phi(a;\nu)\,\Phi(c;\nu)}{\Phi(a+c;\nu)\,[\wp(a)-\wp(c)]} \\ &= 1+\frac{\psi'(c)-\psi'(a)}{(\ln\Phi(c;\nu))'-(\ln\Phi(a;\nu))'} \\ &= 1+\frac{\psi'(c)-\psi'(a)}{\zeta(\nu-a)+\zeta(a)-\zeta(\nu-c)-\zeta(c)}. \end{aligned}$$

Upon taking the $c\to\nu$ limit we obtain

$$e^{\psi(a+\nu)-\psi(a)-\psi(\nu)}=1,$$

which has solution

$$\psi(a)=\mu a.$$

We have thus obtained the Krichever-Höhn elliptic genus

$$g(x)=e^{\mu x}\,\frac{\sigma(\nu-x)}{\sigma(\nu)\,\sigma(x)},$$

so establishing the theorem. □

Remark. We note that (13) is equation [4.3.3] in Hirzebruch [22] which was derived under the hypothesis of invariance under flops; looking at the pole term $c=0$ in equation (13) yields

$$0=g(a)g(a)g(-a)-\left[\alpha_1^2-3\alpha_2\right]g(a)-\alpha_2 g'(a)+\frac{1}{2}g''(a),$$

which is [4.3.2] in Hirzebruch [22].

Remark. The vanishing of λ shows that $\phi(\overline{CP}(3))=0$, yielding an alternative proof to that of [31].

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